EFFECT OF CONVECTION ON THE MOTION OF SPHERICAL BUBBLES

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An expression is derived for the drag coefficient of a spherical bubble, taking account of free, forced, and thermocapillary convection.

We study the effect of free, forced, and thermocapillary convection on the motion of a spherical bubble in a region where the temperature varies linearly with a coordinate. The motion of a spherical bubble, taking account of forced and thermocapillary convection, was treated in [1, 2]. The present article is a refinement and generalization of the results in [3].

Let us consider a spherical bubble of radius R, rising with a velocity U in an infinite incompressible viscous liquid in which a constant temperature gradient dA/dZ is maintained to infinity (Fig. 1). We assume that the Reynolds number $Re_0 \gg 1$ and $Re_1 \gg 1$, and that the conditions in [3] are satisfied.

We seek the velocity distribution in the liquid and vapor phases in the form [3]

$$\vec{u}_0 = \vec{u}_{0V} + \vec{V}_0, \quad \vec{u}_1 = \vec{u}_{1V} + \vec{V}_1, \tag{1}$$

where

$$u_{0,\theta,V} = 1.5U(1 - y/R)\sin\theta; \quad u_{0,r,V} = 1.5Uy/R\cos\theta;$$
$$v_{0,\theta} = UF_0/\sin\theta; \quad v_{1,\theta} = UF_1/\sin\theta; \quad u_{1,\theta,V} = 1.5U(1 + 4y/R)\sin\theta;$$
$$u_{1,r,V} = -3Uy/R\cos\theta.$$

The functions F_0, F_1, $u_{0,r}$, and $u_{1,r}$ satisfy the following system of equations [3]:

$$\frac{\partial F_0}{\partial \psi} = \frac{1}{Re_0}\frac{\partial^2 F_0}{\partial x^2} + \frac{\tilde{\beta}_0 \Delta T}{\sin^2\theta},$$
$$\frac{\partial F_1}{\partial \psi} = \frac{1}{Re_1}\frac{\partial^2 F_1}{\partial x^2} + \frac{\tilde{\beta}_1 \Delta T_1}{\sin^2\theta}, \tag{2}$$

$$\frac{\partial T_0}{\partial \psi} = \frac{1}{Pe_0}\frac{\partial^2 T_0}{\partial x^2}, \quad \frac{\partial T_1}{\partial \psi} = \frac{1}{Pe_1}\frac{\partial^2 T_1}{\partial x^2}, \tag{3}$$

$$\frac{\partial F_0}{\partial \theta} + \sin\theta\frac{\partial u_{0r}}{\partial z} = 0, \quad \frac{\partial F_1}{\partial \theta} + \sin\theta\frac{\partial u_{1r}}{\partial z} = 0 \tag{4}$$

and the boundary conditions

TABLE 1. Dependence of Re_0 on Gr_0 and Fr for Ma = 0 and Ma = 100

Ma=0		Ma=100		
Fr	Re_0	Gr_0	Fr	Re_0
2096	80	0	2026	80
2602	100	0	2300	94
5121	200	10^3	2300	86
7568	300	10^3	2500	98
10412	400	10^3	3000	118
		10^3	5000	180
		10^4	3000	80
		10^4	5000	160

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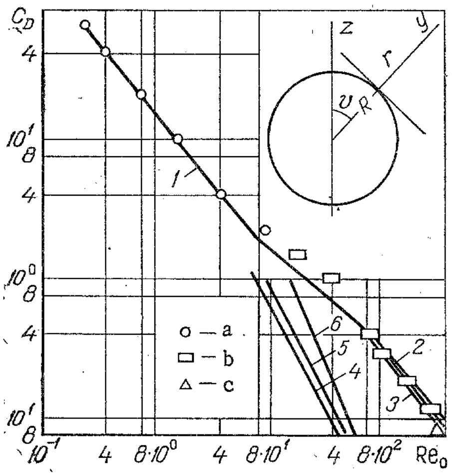

Fig. 1

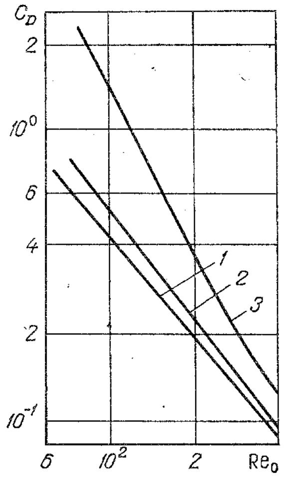

Fig. 2

Fig. 1. Dependence of C_D on Re_0 (curves 1, 2, 3) and C_D on Re_0 and Gr_0 (4, 5, 6); 4) $Gr_0 = 0$; 5) 10^3; 6) 10^4; a, b) data from [8]; c) [9].

Fig. 2. Dependence of C_D on Re_0 and Gr_0 for $Ma = 100$; 1) $Gr_0 = 0$; 2) 10^4; 3) 10^5.

$$F_0|_{x=0} = F_1|_{x=0},\quad F_0(0,\ x) = 0,\quad F_1(0,\ x) = 0,\quad \left.\frac{\partial F_0}{\partial x}\right|_{x=\infty} = 0,$$

$$\left.\frac{\partial F_1}{\partial x}\right|_{x=-\infty} = 0,\quad \mu_0\frac{\partial F_0}{\partial x} - \mu_1\left.\frac{\partial F_1}{\partial x}\right|_{x=0} = 3(\mu_0 + 4\mu_1) + R\,\frac{d\sigma}{dT}\,\frac{dA}{dz}\Big/U - \frac{d\sigma}{dT}\,\frac{1}{\sin\theta}\,\left.\frac{\partial T'}{\partial\theta}/U\right|_{x=0},$$

$$\left.\frac{\partial T_0}{\partial x}\right|_{x=\infty} = \frac{dA}{dz}\cos\theta,\quad \left.\frac{\partial T_1}{\partial x}\right|_{x=\infty} = 0,\quad T_0|_{x=0} = T_1|_{x=0},\quad \lambda_0\left.\frac{\partial T_0}{\partial x}\right|_{x=0} =$$

$$= \lambda_1\frac{\partial T_1}{\partial x},\quad T_0(0,\ x) = 0,\quad T_1(0,\ x) = 0, \tag{5}$$

where $z = y/R$; $x = z\sin^2\theta$; $\psi = \frac{4}{3}\left(\frac{2}{3} - \cos\theta + \frac{\cos^3\theta}{3}\right)$; $\tilde{\beta}_0 = 2Rg\beta_0/U^2$; $\tilde{\beta}_1 = 2Rg\beta_1/U^2$; $T' = T_0 - \frac{dA}{dz}\,r\cos\theta$. Solving problem (3)-(5), we obtain

$$T_0 = -\frac{2R}{\sqrt{Pe_0}}\,\frac{dA}{dz}\left(\int_0^{\psi}\frac{d}{d\lambda}(\cos\theta)\,\mathrm{erf}\left(\frac{x}{2}\sqrt{\frac{Pe_0}{\psi-\lambda}}\right)d\lambda\right.$$

$$\left. + x\int_0^{\psi}\frac{d}{d\lambda}\left(\frac{\cos\theta}{\sin^2\theta}\right)\mathrm{erf}\left(\frac{x}{2}\sqrt{\frac{Pe_0}{\psi-\lambda}}\right)d\lambda\right) + \frac{dA}{dz}(R+y)\cos\theta$$

$$+ \frac{1}{\pi\left(\frac{\lambda_0}{\lambda_1}\sqrt{\frac{a_1}{a_0}} - 1\right)}\left[\int_0^{\psi}\frac{\exp\left(-\frac{x^2 Pe_0}{\psi-\lambda}\right)}{\sqrt{\psi-\lambda}}\,\frac{d}{d\lambda}\int_0^{\lambda}\frac{\varphi_2(\tau)}{\sqrt{\lambda-\tau}}\,d\tau d\lambda\right.$$

$$\left. - \frac{2 - \frac{\lambda_0}{\lambda_1}\sqrt{\frac{a_1}{a_0}}}{\frac{1}{\lambda_1}\sqrt{\frac{a_1}{a_0}} - \frac{1}{\lambda_0}}\int_0^{\psi}\frac{\exp\left(-\frac{x^2 Pe_0}{4(\psi-\lambda)}\right)}{\sqrt{\psi-\lambda}}\,\varphi_3(\lambda)\,d\lambda\right],$$

$$T_1 = \frac{1}{1 - \frac{\lambda_1}{\lambda_0}\sqrt{\frac{a_0}{a_1}}}\left[\frac{1}{\pi}\int_0^{\psi}\frac{\exp\left(-\frac{x^2 Pe_1}{4(\psi-\lambda)}\right)}{\sqrt{\psi-\lambda}}\,\frac{d}{d\lambda}\times\right.$$

$$\times \int_0^{\lambda} \frac{\varphi_2(s)\, ds d\lambda}{\sqrt{\lambda - s}} - \frac{1}{\sqrt{\pi \mathrm{Pe}_0}} \int_0^{\psi} \exp\left(-\frac{x^2 \mathrm{Pe}_1}{4(\psi - \tau)}\right) \frac{\varphi_3(\tau)\, d\tau}{\sqrt{\psi - \tau}} \Bigg], \tag{6}$$

where

$$\varphi_2 = \frac{dA}{dz} R\cos\theta; \quad \varphi_3 = \frac{dA}{dz}\frac{\cos\theta}{\sin\theta} + \frac{dA}{dz}\frac{\sqrt{\mathrm{Pe}_0}}{2}\int_0^{\psi}\int_0^{\infty} \frac{d}{d\lambda}\left(\left(R + \frac{y}{\sin^2\theta}\right)\cos\theta\right) \frac{\exp\left(-\frac{y^2 \mathrm{Pe}_0}{4(\psi-\lambda)}\right) y}{(\psi-\lambda)^{3/2}}\, dy d\lambda.$$

Solving the boundary-value problem (2)-(5) and using (6) and [4], we obtain

$$F_0 = -6\,\frac{\mu_0 + 4\mu_1 + R\frac{d\sigma}{dT}\frac{dA}{dz}\Big/ 3U}{\mu_0\sqrt{\mathrm{Re}_0} + \mu_1\sqrt{\mathrm{Re}_1}}\,\psi^{1/2}\,\mathrm{ierfc}\left(\frac{x}{2}\frac{\sqrt{\mathrm{Re}_0}}{\sqrt{\psi}}\right) + \chi_0 + \frac{1}{\mu_0\sqrt{\pi \mathrm{Re}_0}}\int_0^{\psi} \frac{q_1 + \chi\mu_0}{\sqrt{\psi-\tau}} \exp\left(-\frac{x^2 \mathrm{Pe}_0}{4(\psi-\tau)}\right) d\tau,$$

$$F_1 = -6\,\frac{\mu_0 + 4\mu_1 + R\frac{d\sigma}{dT}\frac{dA}{dz}\Big/ 3U}{\mu_0\sqrt{\mathrm{Re}_0} + \mu_1\sqrt{\mathrm{Re}_1}}\,\psi^{1/2}\,\mathrm{ierfc}\left(\frac{x\sqrt{\mathrm{Re}_1}}{2\sqrt{\psi}}\right) + \chi_1 + \frac{1}{\mu_1\sqrt{\pi \mathrm{Re}_1}}\int_0^{\psi} \frac{q_1}{\sqrt{\psi-\tau}} \exp\left(-\frac{x^2 \mathrm{Pe}_1}{4(\psi-\tau)}\right) d\tau, \tag{7}$$

where

$$\chi_0 = \frac{\mathrm{Gr}_0}{16\sqrt{\pi}\,\mathrm{Re}_0^{3/2}}\int_0^{\psi}\int_0^{\infty} \frac{\Delta T' dy d\lambda}{\sin^2\theta\sqrt{\psi-\lambda}}\left[\exp\left(-\frac{(x-y)^2 \mathrm{Re}_0}{4(\psi-\lambda)}\right) - \exp\left(-\frac{(x+y)^2 \mathrm{Re}_0}{4(\psi-\lambda)}\right)\right];$$

$$\chi_1 = \frac{\mathrm{Gr}_1}{16\sqrt{\pi}\,\mathrm{Re}_1^{3/2}}\int_0^{\psi}\int_0^{\infty} \frac{\Delta T_1' dy d\lambda}{\sin^2\theta\sqrt{\psi-\lambda}}\left[\exp\left(-\frac{(x-y)^2 \mathrm{Re}_1}{4(\psi-\lambda)}\right) - \exp\left(-\frac{(x+y)^2 \mathrm{Re}_1}{4(\psi-\lambda)}\right)\right];$$

$$\chi = \left(\mu_1\frac{\partial\chi_1}{\partial x} - \mu_0\frac{\partial\chi_0}{\partial x}\right)\Bigg|_{x=0} - \frac{4\sin^3\theta}{3\pi U}\left(\frac{1}{\left(\frac{\lambda_0}{\lambda_1}\sqrt{\frac{a_1}{a_0}} - 1\right)}\times\left[\frac{d}{d\psi}\int_0^{\psi}\frac{1}{\sqrt{\psi-\lambda}}\frac{d}{d\lambda}\int_0^{\lambda}\frac{\varphi_2(\tau)\,d\tau}{\sqrt{\lambda-\tau}}\,d\lambda - \frac{2 - \frac{\lambda_0}{\lambda_1}\sqrt{\frac{a_1}{a_0}}}{\frac{1}{\lambda_1}\sqrt{\frac{a_1}{a_0}} - \frac{1}{\lambda_0}}\frac{d}{d\psi}\int_0^{\psi}\frac{\varphi_3(\lambda)\,d\lambda}{\sqrt{\psi-\lambda}}\right]; \quad q_1 = \frac{\chi}{\sqrt{\pi \mathrm{Re}_0}}\left(\frac{1}{\frac{1}{\mu_1\sqrt{\pi \mathrm{Re}_1}}} - \frac{1}{\frac{1}{\mu_0\sqrt{\pi \mathrm{Re}_0}}}\right).$$

We calculate the resisting force D, using [5]:

$$D = 2\pi R^2 \int_0^{\pi} (\tau_{r\theta}\sin\theta - \tau_{rr}\cos\theta)|_{r=R}\sin\theta d\theta, \tag{8}$$

where

$$\tau_{rr} = -p_0 + 2\mu_0\frac{\partial u_{0,r}}{\partial r}; \quad \tau_{r\theta} = \mu_0\left(\frac{1}{r}\frac{\partial u_{0,r}}{\partial\theta} - \frac{u_{0,\theta}}{r}\right).$$

Substituting (1) and (7) into (8) and neglecting quantities of the order $1/\mathrm{Re}_0$, we obtain

$$D = \frac{1.2075\pi R\mu_0 U\,\mathrm{Gr}_0\left(0.254 - \frac{\mu_1}{3\mu_0}\sqrt{\frac{\nu_0}{\nu_1}}\right)}{\mathrm{Re}_0^{3/2}\left(1 - \frac{\mu_1}{\mu_0}\sqrt{\frac{\nu_0}{\nu_1}}\right)}\left(1 + O\left(\frac{1}{\sqrt{\mathrm{Re}_0}}\right)\right) + \frac{8\pi R\mu_0 U}{\left(1 + \left(\frac{\mu_1\rho_1}{\mu_0\rho_0}\right)^{1/2}\right)}\left(1 + \frac{4\mu_1}{\mu_0} + \frac{1}{3}\frac{d\sigma}{dT}\frac{dA}{dz}\Big/\mu_0 U\right)\left(1 + \frac{0.815}{\sqrt{\mathrm{Re}_0}} + O\left(\frac{1}{\sqrt{\mathrm{Re}_0}}\right)\right). \tag{9}$$

Using (9) and neglecting $(\mu_1/\mu_0)\sqrt{\nu_0/\nu_1}$, which is very much less than unity far from the critical point, we calculate the drag coefficient C_D:

$$C_D = \frac{2D}{\rho_0 \pi R^2 U^2} = 1.2256 \frac{\mathrm{Gr}_0}{\mathrm{Re}_0^{5/2}} + \frac{32}{\mathrm{Re}_0}\left(1 + \frac{1}{3}\frac{\mathrm{Ma}}{\mathrm{Re}_0 \mathrm{Pr}}\right)\left(1 + \frac{0.815}{\sqrt{\mathrm{Re}_0}}\right). \tag{10}$$

We consider special cases for (10):

1. $dA/dz = 0$. Then

$$C_D = \frac{32}{\mathrm{Re}_0}\left(1 + \frac{0.815}{\sqrt{\mathrm{Re}_0}}\right). \tag{11}$$

The following expression was derived in [6] for the drag coefficient of a gas bubble in a liquid for small and moderate Reynolds numbers ($0 < \mathrm{Re}_0 \leqslant 5$):

$$C_D = \frac{16}{\mathrm{Re}_0}(1 + 0.125\mathrm{Re}_0). \tag{12}$$

Since, according to [5], the tangential stresses vanish on the surface of a bubble, there is nonseparating flow around a bubble. Because of nonseparating flow around a spherical bubble the drag coefficient of the bubble is a monotonically decreasing function of the Reynolds number.

By using a dissipation function and asymptotic boundary layer methods the following expression was derived in [7] for the drag coefficient of a spherical gas bubble at large Reynolds numbers:

$$C_D = \frac{48}{\mathrm{Re}_0}\left(1 - \frac{2.2}{\sqrt{\mathrm{Re}_0}}\right). \tag{13}$$

By calculating the dependence of C_D and Re_0 from Eq. (12) in the range $0 < \mathrm{Re}_0 \leqslant 5$, and by Eq. (13) for $70 \leqslant \mathrm{Re}_0 \leqslant 350$, and interpolating graphically for intermediate Reynolds numbers, we obtain a unique dependence of C_D on Re_0 for $0 < \mathrm{Re}_0 \leqslant 350$ (Fig. 1, curve 1). Curve 2 shows the dependence of C_D on Re_0 for $70 \leqslant \mathrm{Re} \leqslant 350$ from Eq. (11). The maximum difference of the results calculated by Eqs. (11) and (13) is 20%. For comparison Fig. 1 also shows the experimental dependence of C_D on Re_0 [8, 9]. Curve 1 was approximated to within 1% by the method of least squares:

$$\ln C_D = 2.77014 - 2.45195\log_{10}\mathrm{Re}_0 + 0.0820358(\log_{10}\mathrm{Re}_0)^2 +$$
$$+ 0.338871(\log_{10}\mathrm{Re}_0)^3 + 0.109637(\log_{10}\mathrm{Re}_0)^4 - 0.155004(\log_{10}\mathrm{Re}_0)^5 + 0.0301038(\log_{10}\mathrm{Re}_0)^6. \tag{14}$$

Curve 3 shows C_D as a function of Re_0 calculated by Eq. (14). The calculated values of the drag coefficients (1, 2, and 3) are in good agreement with the experimental data.

2. Case of Weightlessness ($g = 0$). We obtain from (10)

$$C_D = \frac{32}{\mathrm{Re}_0}\left(1 + \frac{\mathrm{Ma}}{3\mathrm{Re}_0\mathrm{Pr}}\right)\left(1 + \frac{0.815}{\sqrt{\mathrm{Re}_0}}\right). \tag{15}$$

Curve 4 shows the calculated dependence of C_D on Re_0 for $\mathrm{Ma} = 0$ (neglecting the Marangoni effect) for $80 < \mathrm{Re}_0 \leqslant 350$ and $0 < \mathrm{Gr}_0 \leqslant 10^5$.

Figure 2 shows the dependence of C_D on Re_0 and Gr_0 for $\mathrm{Ma} = 100$ ($80 \leqslant \mathrm{Re}_0 \leqslant 350$), ($0 \leqslant \mathrm{Gr}_0 \leqslant 10^5$).

It follows from Figs. 1 (4) and 2 that C_D is a monotonically increasing function of Ma and Gr_0, and for $\mathrm{Gr}_0 \geqslant 10^3$ it is necessary to take account of the effect of free convection on the drag coefficient C_D.

Equating the resisting force to the sum of the Archimedes and thermocapillary forces, we obtain the following equation for the velocity of rise of a bubble:

$$C_D = \frac{4}{3}\frac{\mathrm{Fr}}{\mathrm{Re}_0^2} + 4\pi\frac{\mathrm{Ma}}{\mathrm{Pr}\mathrm{Re}_0^2}.$$

Table 1 shows the dependence of Re_0 on Fr and Gr_0 for $\mathrm{Ma} = 0$ and 100.

NOTATION

R, radius of bubble; r, θ, spherical coordinates; v_r, v_θ, radial and longitudinal velocity components; z, vertical coordinate; U, velocity of rise of bubble; p_0, p_1, pressure; β_0, β_1, coefficient of volume expansion; ρ_0, ρ_1, density; μ_0, μ_1, dynamic viscosity; a_0, a_1, thermal diffusivity of liquid and vapor, respectively; σ, surface tension; $\tau_{r\theta}$, τ_{rr}, tangential and normal stresses; D, resisting force; E(x, r), F(x, r), elliptic integrals of the first and second kinds; $z = y/R$; $x = z\sin^2\theta$; $v_0 = v_{0,\theta}/U$; $v_1 = v_{1\theta}/U$; $F_i = v_i \sin\theta$; $\nu_i = \mu_i/\rho_i$; $C_D = 2D/(\pi\rho_0 U^2 R^2)$; $Re_i = 2RU/\nu_i$; $Pr_i = \nu_i/a_i$; $Pe_i = Re_i Pr_i$; $Gr_i = g\beta_i \times (dA/dz)R^4/\nu_i^2$; $Fr = 8gR^3/v_0^2$; $Ma = 2\frac{d\sigma}{dT}\frac{dA}{dz}R/v_0 a_0$; $\Phi(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-s^2)\,ds$; $\Phi^* = 1 - \Phi$; $i\Phi^*(x) = \int_x^\infty \Phi^*(s)\,ds$. Subscript i = 0.1.

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FREE CONVECTION IN A GRAINY LAYER ALONG A VERTICAL WALL

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A solution based on the integral thermal balance equation is offered.

We propose an approximate analytical solution of the problem of free convection produced by the temperature difference between a wall and a liquid filling an immobile grainy layer of solid elements. The solution obtained is also applicable to the process of mass exchange.

We make the following assumptions in considering the problem.

1. Liquid convection in the layer occurs in the region of dominance of viscosity forces.

2. The temperature difference in the layer is not large, so that the physical parameters of the liquid (aside from density) are temperature independent; the density is a linear function of temperature.

3. The temperatures of grains and liquid are identical, i.e., the layer is considered as a quasihomogeneous medium [1, p. 103].

4. Thermal conductivity in the layer along the liquid flow and thermal resistance at the wall [1, p. 127] are neglected.

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